600.271 Automata & Computation Theory
Second Mid-Semester Examination
November 10, 2009
In-class, Closed Book
Time: 1 hr, 15 minutes

I. (10 pts.) Design a context-free grammar for the language
$\{a^i b^j c^k \mid i, j, k \geq 1,\ (i \neq j) \text{ or } (j = k \text{ and } i + j + k \text{ is a multiple of } 3)\}$

II. (10 pts.) Establish the decidability of one of the following problems by designing an appropriate algorithm.

- Given a CFG G and 2 positive integers ℓ and h, $\ell < h$, is there an $x \in L(G)$ s.t. $\ell \leq |x| \leq h$?
- Given $[M]$ and x, does dTM M halt on input x while utilizing no more than $|x|^2$ space?

III. (10 pts.) Design a Turing machine for computing the following function.

$$f(x, y, z) = (y, x + 2z).$$

IV. (10 pts.) Solve one of the following problems.

- Prove that the problem of given $[M]$, to test whether for every $n \geq 1$ the dTM M halts on at least $n/4$ inputs x, $x \leq n$, and at most $n/2$ inputs x, $x \leq n$, is undecidable. (Hint: This problem is quite easy.)

  In a PCP_1 problem, given (x_i, y_i), for $i = 1, 2, \cdots, n$, and a value k, to test whether there exist $i_1, i_2, \cdots, i_m$ s.t. $x_{i_1} x_{i_2} \cdots x_{i_m} = y_{i_1} y_{i_2} \cdots y_{i_m}$, and $k \in \{i_1, i_2, \cdots, i_m\}$. In a PCP_{all} problem, given (x_i, y_i), for $i = 1, 2, \cdots, n$, to test whether there exist $i_1, i_2, \cdots, i_m$ s.t. $x_{i_1} x_{i_2} \cdots x_{i_m} = y_{i_1} y_{i_2} \cdots y_{i_m}$, and every k, $1 \leq k \leq n$, is in $\{i_1, i_2, \cdots, i_m\}$.

- Prove that $PCP_{all} \leq_T PCP_1$.

- Prove that $PCP_1 \leq_m PCP_{all}$. (Hint: This problem is hard.)