

CS 461 - Computer Vision  
 Professor Greg Hager  
 Fall 2006 Homework 2 - Answer Key

1.)

a.) From the definition of convolution we have:

$$g(x) = (f * K_1)(x) = \int f(u)K_1(x - u)du$$

Similarly convolving  $g$  with  $K_2$  we have:

$$(g * K_2) = (g * K_2)(x) = \int g(v)K_2(x - v)dv$$

Substituting

$$(f * K_1 * K_2)(x) = \iint (f(u)K_1(x - u)du)K_2(x - v)dv$$

b.)

Given  $G(x)$  is  $N(0, x)$ .  $G(x) = \frac{1}{\sqrt{2\pi x}} e^{-\frac{t^2}{2x}}$

$$G(y) = G(x) * G(x) = \int \frac{1}{\sqrt{2\pi x}} e^{-\frac{u^2}{2x}} \frac{1}{\sqrt{2\pi x}} e^{-\frac{(t-u)^2}{2x}} du$$

$$\begin{aligned} &= \frac{1}{2\pi x} \int e^{-\frac{u^2}{2x}} e^{-\frac{(t-u)^2}{2x}} du \\ &= \frac{1}{2\pi x} \int e^{-\frac{u^2}{2x}} e^{-\frac{t^2+2tu-u^2}{2x}} du \\ &= \frac{1}{2\pi x} e^{-\frac{t^2}{2x}} \int e^{-\frac{u^2}{2x}} e^{-\frac{2tu-u^2}{2x}} du \\ &= \frac{1}{2\pi x} e^{-\frac{t^2}{2x}} \int e^{-\frac{u^2}{x}} e^{-\frac{tu}{x}} du \\ &= \frac{1}{2\pi x} e^{-\frac{t^2}{2x}} \int e^{-\frac{u^2}{x}} e^{-\frac{tu}{x}} du \\ &= \frac{1}{2\pi x} e^{-\frac{t^2}{2x}} \int e^{-\frac{u^2+tu}{x}} du \\ &= \frac{1}{2\pi x} e^{-\frac{t^2}{4x}} \int e^{-\frac{(u+\frac{1}{2}t)^2}{x}} du \end{aligned}$$

Substituting

$$\begin{aligned} z &= (u - (1/2)t)^2 \quad dz = du \\ &= \frac{1}{2\pi x} e^{-\frac{t^2}{4x}} \int e^{-\frac{z^2}{x}} dz = \frac{1}{2\pi x} e^{-\frac{t^2}{4x}} \sqrt{\pi x} \\ &= \frac{1}{\sqrt{4\pi x}} e^{-\frac{t^2}{4x}} \end{aligned}$$

Thus  $y = 2x$

2.) a.)

From the notes we have

$$I_G = I * H(x) = \sum_{h=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} H(h, k) I(i - h, j - k)$$

Thus,

$$I_G = \sum_{h=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} e^{-\frac{h^2+k^2}{2x}} I(i-h, j-k)$$

b.) From 2a we can write it because summation is linear and the properties of  $e$

$$\begin{aligned} I_G &= \sum_{h=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} e^{-\frac{h^2+k^2}{2x}} I(i-h, j-k) \\ &= \sum_{h=-\infty}^{\infty} e^{-\frac{h^2}{2x}} \sum_{k=-\infty}^{\infty} e^{-\frac{k^2}{2x}} I(i-h, j-k) \end{aligned}$$

Forming

$$G(x_1) \text{ and } G(x_2) * I$$

Thus

$$I_G = I * H(x) = G(x_1) * G(x_2) * I$$

From commutativity

$$I_G = I * H(x) = I * G(x_1) * G(x_2)$$

Thus the Gaussian kernel is separable.

3.) From the definition of

$$(f * g)(x) = \int_{-\infty}^{+\infty} f(u)g(x-u)du$$

substituting

$z = x - u$      $dz = -dz$     and the limit are now reversed

$$\begin{aligned} (f * g)(x) &= \int_{-\infty}^{+\infty} f(x-z)g(z)du \\ &= -\int_{+\infty}^{-\infty} f(x-z)g(z)du \\ &= \int_{-\infty}^{+\infty} f(x-z)g(z)du \\ &= \int_{-\infty}^{+\infty} g(z)f(x-z)du = (g * f)(x) \end{aligned}$$

Thus  $(f * g)(x) = (g * f)(x)$ , proving convolution is commutative.

4.)

First, let's look in the continuous domain. Suppose  $n$  is the period of our underlying sin wave signal (e.g.  $n$  is in pixels). In this case, our signal is

$$S_n(u) = \sin(2\pi u/n)$$

The result of convolution is:

$$C_n(x) = \frac{1}{2m} \int_{-m}^m S_n(x-y)dy = \frac{1}{2m} \int_{-m}^m \sin(2\pi(x+y)/n)dy$$

The indefinite integral is:

$$\int \sin(2\pi(x+y)/n)dy = -n \cos(2\pi(x+y)/n)/(2\pi)$$

Plugging in, we arrive at

$$\begin{aligned} C_n(x) &= \frac{n}{4\pi m} (-\cos(2\pi(x+m)/n) - (-\cos(2\pi(x-m)/n))) \\ &= \frac{n}{4\pi m} (\cos(2\pi(x-m)/n) - \cos(2\pi(x+m)/n)) \end{aligned}$$

Suppose I choose  $2m = kn$ , or equivalently  $m = kn/2$  for some integer  $k$ . Then this becomes

$$\begin{aligned} C_n(x) &= \frac{n}{4\pi kn/2} \cos(2\pi(x - kn/2)/n) - \cos((x + kn/2)/n) \\ &= \frac{1}{2\pi k} (\cos(2\pi x - k\pi) - \cos(2\pi x + k\pi)) = 0 \end{aligned}$$

The latter follows since  $\cos$  is periodic with period  $2\pi$  and the fact that  $k$  is an integer. Thus, for a given  $m$ , any signal with period  $n = 2m/k$  will yield a zero output.

We can also see that choosing  $2m = kn + n/2 = n(k + 1/2)$  yields

$$\begin{aligned} C_n(x) &= \frac{n}{4\pi((kn/2) + n/4)} \cos(2\pi(x - n(k + 1/2)/2)/n) - \cos((x + n(k + 1/2)/2)/n) \\ &= \frac{1}{2\pi k + \pi} (\cos(2\pi x - k\pi - \pi/2) - \cos(2\pi x + k\pi + \pi/2)) \\ &= \frac{2}{2\pi k + \pi} (\cos(2\pi x - k\pi - \pi/2)) \end{aligned}$$

By inspection, this is a local maximum (with respect to choice of  $m$ ) of output since any other choice of  $k$  would clearly yield less signal (this could be proven formally setting up the difference of the output over  $1/2$  period and then taking derivatives with respect to  $k$ .) Note that the output attenuates with larger values of  $k$ .

In the discrete domain, the equivalent is to average with a box filter of size  $2m + 1$ . The minima and maxima occur for values of  $\sin$  that have periods that satisfy  $kn = (2m + 1) + 1 = 2(m + 1)$  for integral values of  $k$ . Equivalently, the wavelength is  $2(m + 1)/k$ . The extra  $+1$  in this case is to cover for the fact that the discrete filter should not completely cover to the start of the next period (but this is a minor point). Likewise, the maximum output is at  $(k + 1/2)n = (2m + 1) + 1 = 2(m + 1)$ .