CS 461 - Computer Vision
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Fall 2006 Homework 3 - Answer Key

1.) A homogeneous transform is represented as:

$$F = \begin{bmatrix} R & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}$$

We want to find F' such that $FF' = I_{4\times4}$. Thus $F' = F^{-1}$

Then

$$\begin{bmatrix} R & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}\begin{bmatrix} R' & \mathbf{t}' \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} RR' & R\mathbf{t}' + \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} = I_{4\times4}$$

Thus

$RR' = I_{3\times3}$

$R\mathbf{t}' + \mathbf{t} = \mathbf{0}$

Solving for R' and $\mathbf{t}'$

$R' = R^{\intercal}$

$\mathbf{t}' = -R^{\intercal}\mathbf{t}$

Therefore,

$$F^{-1} = \begin{bmatrix} R^{\intercal} & -R^{\intercal}\mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}$$

2.) Rotation of $\theta = 30^{\circ}$ about axis $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^{\intercal}$

Unit vector direction is $\hat{n} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}^{\intercal}$

A 3D rotation can be expressed as

$$R = I\cos\theta + (1 - \cos\theta)\hat{n}\hat{n}^{\intercal} + \sin\theta(sk(\hat{n}))$$

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\frac{\sqrt{3}}{2} + (1 - \frac{\sqrt{3}}{2})\begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \end{bmatrix}$$

$$R = \begin{bmatrix} 0.91068 & -0.24402 & 0.33333 \\ 0.33333 & 0.91068 & -0.24402 \\ -0.24402 & 0.33333 & 0.91068 \end{bmatrix}$$

3.) a.)

$$\begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 1 \end{pmatrix} \left[\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right] = 0$$

$$\begin{pmatrix} \frac{-1}{2} \\ \frac{\sqrt{3}}{2} \\ 1 \end{pmatrix} \left[\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right] = 0$$

b.) $\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x' \\ y' \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{-1}{2} \\ \frac{\sqrt{3}}{2} \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x' \\ y' - 2 \\ 0 \end{bmatrix} = 0$

$$-\frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}y' = -\frac{1}{2}x' + \frac{\sqrt{3}}{2}y' - \sqrt{3} = 0$$

$$-\frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}y' = 0 \qquad \Rightarrow x' = y'$$

Substituting

$-\frac{1}{2}x' + \frac{\sqrt{3}}{2}x' - \sqrt{3} = 0$, Solution is: $x' = \frac{\sqrt{3}}{\frac{1}{2}\sqrt{3} - \frac{1}{2}} = \sqrt{3} + 3$

Thus

$$\begin{pmatrix} \sqrt{3} + 3 \\ \sqrt{3} + 3 \\ 1 \end{pmatrix}$$

4.)

First line

$$\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} 100 \\ 100 \\ 1000 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

Second line

$$\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} 200 \\ 200 \\ 1100 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

First start by finding two points on the 3D lines and their respective projections.

Two point from the first line are

$$\begin{pmatrix} 100 \\ 100 \\ 1000 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 200 \\ 300 \\ 1100 \\ 1 \end{pmatrix}$$

Then from the second line

$$\begin{pmatrix} 200 \\ 200 \\ 1100 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 300 \\ 400 \\ 1200 \\ 1 \end{pmatrix}$$

Using the following projection we find projection of the 3D homogenous coordinate to 2D

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{pmatrix} 100 \\ 100 \\ 1000 \end{pmatrix} \text{ and } \begin{pmatrix} 200 \\ 300 \\ 1100 \end{pmatrix} \qquad \begin{pmatrix} 200 \\ 200 \\ 1100 \end{pmatrix} \text{ and } \begin{pmatrix} 300 \\ 400 \\ 1200 \end{pmatrix}$$

Normalizing the Homogenous coordinates yields

$$\frac{1}{1000}\begin{pmatrix}100\\100\\1000\end{pmatrix}=\begin{pmatrix}\frac{1}{10}\\\frac{1}{10}\\1\end{pmatrix}\text{ and }\frac{1}{1100}\begin{pmatrix}200\\300\\1100\end{pmatrix}=\begin{pmatrix}\frac{2}{11}\\\frac{3}{11}\\1\end{pmatrix}$$

$$\frac{1}{1100}\begin{pmatrix}200\\200\\1100\end{pmatrix}=\begin{pmatrix}\frac{2}{11}\\\frac{2}{11}\\1\end{pmatrix}\text{ and }\frac{1}{1200}\begin{pmatrix}300\\400\\1200\end{pmatrix}=\begin{pmatrix}\frac{1}{4}\\\frac{1}{3}\\1\end{pmatrix}$$

Then the project of the first line is

$$\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}\frac{1}{10}\\\frac{1}{10}\end{pmatrix}+\overline{\lambda}\begin{pmatrix}\frac{2}{11}-\frac{1}{10}\\\frac{3}{11}-\frac{1}{10}\end{pmatrix}=\begin{pmatrix}\frac{1}{10}\\\frac{1}{10}\end{pmatrix}+\overline{\lambda}\begin{pmatrix}\frac{9}{110}\\\frac{19}{110}\end{pmatrix}$$

and the second line is

$$\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}\frac{2}{11}\\\frac{2}{11}\end{pmatrix}+\overline{\mu}\begin{pmatrix}\frac{1}{4}-\frac{2}{11}\\\frac{1}{3}-\frac{2}{11}\end{pmatrix}=\begin{pmatrix}\frac{2}{11}\\\frac{2}{11}\end{pmatrix}+\overline{\mu}\begin{pmatrix}\frac{3}{44}\\\frac{5}{33}\end{pmatrix}$$

We want to find the intersection, which is finding the $\overline{\lambda}$ and $\overline{\mu}$ where the two lines are equal.

$$\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}\frac{2}{11}\\\frac{2}{11}\end{pmatrix}+\overline{\mu}\begin{pmatrix}\frac{3}{44}\\\frac{5}{33}\end{pmatrix}=\begin{pmatrix}\frac{1}{10}\\\frac{1}{10}\end{pmatrix}+\overline{\lambda}\begin{pmatrix}\frac{9}{110}\\\frac{19}{110}\end{pmatrix}$$

We have two equation and 2 unknown

$$\begin{aligned}\frac{3}{44}\overline{\mu}+\frac{2}{11}&=\frac{9}{110}\overline{\lambda}+\frac{1}{10}\\\frac{5}{33}\overline{\mu}+\frac{2}{11}&=\frac{19}{110}\overline{\lambda}+\frac{1}{10}\end{aligned}$$

$$\begin{aligned}\frac{3}{44}\overline{\mu}-\frac{9}{110}\overline{\lambda}&=-\frac{9}{110}\\\frac{5}{33}\overline{\mu}-\frac{19}{110}\overline{\lambda}&=-\frac{9}{110}\end{aligned}$$

$$\begin{bmatrix}\frac{3}{44}&-\frac{9}{110}\\\frac{5}{33}&-\frac{19}{110}\end{bmatrix}^{-1}\begin{bmatrix}-\frac{9}{110}\\-\frac{9}{110}\end{bmatrix}=\begin{bmatrix}12\\11\end{bmatrix}=\begin{bmatrix}\overline{\mu}\\\overline{\lambda}\end{bmatrix}$$

$$\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}\frac{2}{11}\\\frac{2}{11}\end{pmatrix}+12\begin{pmatrix}\frac{3}{44}\\\frac{5}{33}\end{pmatrix}=\begin{pmatrix}1\\2\end{pmatrix}$$

5.)We know that we need to find a line with at least n point close to it.   Next the

probability of a certain point being an inlier of the line is $\frac{n}{N}$ where N is the total number of points.
Since these two events are independent then the probability of two being inlier is $(\frac{n}{N})(\frac{n}{N}) = (\frac{n}{N})^2$.
Then we need the probability of a sample of two points taken not being composed of all inliers which is $1 - (\frac{n}{N})^2$.
We then choose k of there independent events $\left(1 - (\frac{n}{N})^2\right)^k$. Final we want the probability that no more lines to exist which is $1 - 0.999 = 0.001$.
Thus
$$\left(1 - (\frac{n}{N})^2\right)^k = 0.001$$
Solving for k
$$k = \frac{\log(0.001)}{\log\left(1 - (\frac{n}{N})^2\right)}$$